Unitary groups

Matevž Miščič

These notes are based on Lecture 5 of Nick Gill's course on Finite Classical Groups [1].

1 Unitary groups

Let β be a non-degenerate σ-Hermitian form on a n dimensional vector space V over a finite field $\mathbf{F}_{q^2}$. Throughout this lecture we will write $\mathbf{F} = \mathbf{F}_{q^2}$ and $\mathbf{F}_0$ for its unique subfield of order q. Recall the *trace* and *norm* maps from F onto F_0 defined by

$$\operatorname{tr}(\lambda) = \lambda + \lambda^q, \qquad \mathrm{N}(\lambda) = \lambda^{q+1}.$$

Recall also that we have a unitary basis of V:

- $B = \{v_1, w_1, \ldots, v_r, w_r\}$ for n even.
- $B = \{v_1, w_1, \ldots, v_r, w_r, u\}$ for n odd.

Here (v_i, w_i) are pairwise orthogonal hyperbolic pairs and u is a vector with $\beta(u, u) = 1$ that is orthogonal to all hyperbolic pairs.

There is also a orthonormal basis of V, which is often useful.

Lemma 1.1. *Vector space V has a basis $\{v_1, \ldots, v_n\}$ with $\beta(v_i, v_j) = \delta_{ij}$.*

Proof. There exists a vector v_1 with $\beta(v_1, v_1) \neq 0$. Since the norm is surjective onto $\mathbf{F}_0$ and $\beta(v_1, v_1) \in \mathbf{F}_0$, we can scale it so that $\beta(v_1, v_1) = 1$. Now $v_1^\perp$ is $(n-1)$-dimensional vector space with a non-degenerate σ-Hermitian form, so we are done by induction. $\square$

Let us define the unitary groups.

Definition 1.2. The groups $\mathrm{GU}_n(q)$ is the isometry group of β and the groups $\mathrm{SU}_n(q)$ is the special isometry group of β.

There are also the projective versions of these groups, which are defined as quotiens by intersection with the subgroup of all scalar transformations.

Proposition 1.3. *We have $\mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$ and under this isomorphism transvections in one group correspond to transvections in the other. In addition the action of $\mathrm{SU}_2(q)$ on its polar space is equivalent to the action of $\mathrm{SL}_2(q)$ on its projective space.*

2 The action of unitary group on its polar space

The group $\mathrm{SU}(V)$ acts naturally on its polar space.

Proposition 2.1. *The kernel of the action of* $\mathrm{SU}(V)$ *on its polar space is the subgroup of all scalar transformations. Moreover, this action is primitive.*

Proof. Let $g \in \mathrm{SU}(V)$ be an element that acts trivially.

Let (v_1, w_1) for $i = 1, \ldots, r$ be pairwise orthogonal hyperbolic pairs. All vectors v_i and w_i are isotropic, so they only get scaled by some scalars. Since $v_i + v_j$ are also isotropic, all vector v_i get scaled by the same scalar. Similarly holds for all w_i. By taking some nonzero λ with $\mathrm{tr}(\lambda) = 0$, we have that $v_1 + \lambda w_1$ is also isotropic, so all vectors v_i and w_i get scaled by the same scalar.

If n is odd, there is also a basis vector u. Since g preserves orthogonality and u up to scalar the only vector that is orthogonal to all v_i and w_i, it also gets scaled by some scalar. It is easy to find μ so that $u + v_1 + \mu w_1$ is isotropic, so u also gets scaled by the same scalar. We conclude that g is a scalar transformation.

We claim that $\mathrm{SU}(V)$ acts transitively on the set of points of the polar space. Indeed, an isometry between two points extends to an isometry of the whole space by Witt's lemma. One then needs to adjust the determinant in the following way. Let v be a nonzero vector and extend it to a hyperbolic pair (v, w). Let g be the map with $gv = \lambda v$ and $gw = \mu w$ that is the identity on $\langle v, w\rangle^{\perp}$. Then g is an isometry if and only if $1 = \beta(v, w) = \beta(gv, gw) = \lambda\mu^{\sigma}$. That is equivalent to $\lambda = \mu^{-q}$. The determinant of g is $\det(g) = \lambda\mu = \mu^{1-q}$, so we can adjust the determinant.

Suppose we have two pairs of points $(\langle v_i, w_i\rangle)$ for $i = 1, 2$ such that $\beta(v_i, w_i) = 1$. By Witt's lemma there is an isometry g with $gv_1 = v_2$ and $gw_1 = w_2$. We can adjust the determinant of g as above, so $\mathrm{SU}(V)$ acts transitively on the set of pairs of points that span a hyperbolic line.

Suppose now that the pairs of points satisfy $\beta(v_i, w_i) = 0$ for $i = 1, 2$. Since $\mathrm{SU}(V)$ is transitive on points, we can assume that $v_1 = v_2$. There are two cases to consider.

- If $\beta(w_1, w_2) \neq 0$ we can scale w_2 so that $\beta(w_1, w_2) = 1$. Then we can adjust the determinant of the isometry of V that swaps w_1 and w_2 and is the identity on $\langle w_1, w_2\rangle^{\perp}$, so that it is an element of $\mathrm{SU}(V)$.

- If $\beta(w_1, w_2) = 0$, we can go through another vector by applying the previous case twice. We will not construct such a vector.

So the permutation rank of the action of $\mathrm{SU}(V)$ on its polar space is at most 3 and now the primitivity follows by the same argument as for the symplectic case. □

3 Unitary transvections

Definition 3.1. A *unitary transvection* of V is a transvection that preserves the Hermitian form β.

Lemma 3.2. *A transvection t is unitary if and only if it is of the form $t : x \mapsto x + \lambda\beta(x, a)a$ for some isotropic vector a and some $\lambda \in F$ with* $\mathrm{tr}(\lambda) = 0$.

Proof. Let t be a transvection. Then there are $a \in V$ and $\phi \in V^*$ with $\phi(a) = 0$ such that $t : x \mapsto x + \phi(x)a$. Assume that t is unitary. Then

$$\begin{aligned}\beta(x, y) &= \beta(tx, ty) = \beta(x + \phi(x)a, y + \phi(y)a) \\ &= \beta(x, y) + \phi(x)\beta(y, a)^\sigma + \phi(y)^\sigma\beta(x, a) + \phi(x)\phi(y)^\sigma\beta(a, a).\end{aligned}$$

By taking $y = a$ we get $\phi(x)\beta(a, a) = 0$ for all x, so $\beta(a, a) = 0$. By taking x such that $\beta(x, a) = -1$ and $y = x$, we get $\mathrm{tr}(\lambda) = 0$ for $\lambda = \phi(x)$. The converse is an easy calculation. $\square$

Let $H = \langle t \in \mathrm{SU}_n(q) \mid t \text{ is a transvection}\rangle$ and $\Gamma = \{v \in V \mid \beta(v, v) = 1\}$.

Lemma 3.3. *The group H acts transitively on Γ unless* $(n, q) = (3, 2)$.

Proof. Assume $n \geq 4$. Let $u, v \in \Gamma$. Suppose that $\beta(u, v) = 0$. Then $\{u, v\}$ is an orthonormal basis for $W = \langle u, v\rangle$. The group $\mathrm{SU}(W)$ is generated by unitary transvections that can be extended by the identity on $W^\perp$ to unitary transvections of V. The linear map on W defined by $u \mapsto v$ and $v \mapsto -u$ preserves the unitary form and has determinant 1, so it lies in $\mathrm{SU}(W)$. Because it is a product of transvections on W, its extension h is a product of transvections on V, so $h \in H$ and $hu = v$.

Suppose that $\beta(u, v) \neq 0$. We will find a vector $w \in \Gamma$ with $\beta(u, w) = 0$ and $\beta(w, v) = 0$. Let $W = \langle u, v\rangle$. Since β is non-degenerate, we have $\dim W^\perp \geq 2$, and since W is not totally isotropic, we have $\dim(W \cap W^\perp) \leq 1$. That means $W^\perp$ is not totally isotropic, so there is a vector $w \in W^\perp$ with $\beta(w, w) \neq 0$. We can scale w so that $\beta(w, w) = 1$. Then w is the desired vector and we can use the previous case twice.

We are left with the case $n = 3$ and $q \geq 3$. We leave it to the reader. $\square$

Proposition 3.4. *The group $\mathrm{SU}_n(q)$ is generated by unitary transvections unless $(n, q) = (3, 2)$.*

Proof. We will prove this by induction on n. The base case for $n = 2$ is given by Proposition 1.3. The case $(n, q) = (4, 2)$ requires special care and is left to the reader.

Now assume $n \geq 3$ and $(n, q) \neq (4, 2)$. Let $g \in \mathrm{SU}_n(q)$ and let $v \in \Gamma$. Since g preserves the unitary form, $gv \in \Gamma$. By the previous lemma, there is an element $h \in H$ such that $hv = gv$.

The element $h^{-1}g$ fixes v, so it preserves the orthogonal complement $v^{\perp}$. Because the determinant of $h^{-1}g$ on $\langle v\rangle$ is 1, its restriction to $v^{\perp}$ must also have determinant 1. Thus, $h^{-1}g$ induces an element of $\mathrm{SU}(v^{\perp}) \cong \mathrm{SU}_{n-1}(q)$. By inductive hypothesis, $h^{-1}g$ is a product of transvections on $v^{\perp}$. Extending these transvections to V by the identity on v, we see that $h^{-1}g \in H$ and thus $g \in H$. ☐

Proposition 3.5. *The group* $\mathrm{SU}_n(q)$ *is perfect unless* $(n,q) \in \{(2,2),(2,3)\}$.

Proof. By the previous proposition it is enough to prove that unitary transvections are commutators. Proposition 1.3 shows that this is true for $n = 2$, so assume $n \geq 3$.

First, assume $q \geq 3$. Let $t\colon x \mapsto x + \lambda\beta(x,a)a$ be a unitary transvection, where a is isotropic and $\mathrm{tr}(\lambda) = 0$. We will express t as the commutator of some other transvection s and an element $h \in \mathrm{SU}_n(q)$.

Let $s(x) = x + \mu\beta(x,a)a$ be a unitary transvection and $h \in \mathrm{SU}_n(q)$ with $ha = \alpha a$. Their conjugate is given by

$$\begin{aligned}(hsh^{-1})(x) &= h\Big(h^{-1}x + \mu\beta(h^{-1}x,a)a\Big)\\ &= x + \mu\beta(h^{-1}x,a)h(a)\\ &= x + \mu\beta(x,h(a))h(a)\\ &= x + \mu\alpha^{q+1}\beta(x,a)a.\end{aligned}$$

It follows that $[s,h](x) = x + \mu(\alpha^{q+1}-1)\beta(x,a)a$. To ensure $[s,h] = t$, we need $\mu(\alpha^{q+1}-1) = \lambda$. The equation $\alpha^{q+1} = 1$ only has $q+1$ solutions, but there are $q^2 - 1$ non-zero elements in $\mathbf{F}_{q^2}$. As $q \geq 3$ we can choose some α with $\alpha^{q+1} \neq 1$ and define $\mu = (\alpha^{q+1}-1)^{-1}\lambda$ Note that $\mathrm{tr}(\mu) = 0$.

It remains to show that an element $h \in \mathrm{SU}_n(q)$ satisfying $ha = \alpha a$ actually exists. We can complete a to a hyperbolic pair (a,b). Define h on $W = \langle a,b\rangle$ by $ha = \alpha a$ and $hb = \alpha^{-q}b$. By Witt's lemma, we can extend h to the entire space. Since $\dim W^{\perp} \geq 1$, we can adjust the determinant of h and keep $ha = \alpha a$. ☐

4 The point stabilizer

To apply Iwasawa's criterion, we still need to find an abelian normal subgroup of the point stabilizer, such that its normal closure in $\mathrm{SU}_n(q)$ is the whole group.

Lemma 4.1. *The point stabilizer of* $\mathrm{SU}_n(q)$ *contains an abelian normal subgroup* Q *isomorphic to the additive group of* $\mathbf{F}$*, such that the normal closure of* Q *in* $\mathrm{SU}_n(q)$ *is the whole group.*

Proof. Let $\langle v\rangle$ be a point in the polar space, where v is an isotropic vector. Consider the group Q of all unitary transvections of the form $x \mapsto x +$

$\lambda\beta(x,v)v$ for $\lambda \in \mathbf{F}$ with $\mathrm{tr}(\lambda) = 0$. This group isomorphic to a subgroup of additive group of $\mathbf{F}$. Take a transvection $t \in Q$ and an element $g \in \mathrm{SU}_n(q)$ with $gv = \alpha v$. Then

$$\begin{aligned} gtg^{-1}(x) &= g(g^{-1}x + \lambda\beta(g^{-1}x, v)v) \\ &= x + \lambda\beta(x, gv)gv \\ &= x + \lambda\alpha^{q+1}\beta(x, v)v \in Q, \end{aligned}$$

so Q is normal in the point stabilizer.

Let t be any transvection in $\mathrm{SU}_n(q)$. By Lemma 3.2 we have $t\colon x \mapsto x + \lambda\beta(x,a)a$ for some isotropic vector a and some $\lambda \in F$ with $\mathrm{tr}(\lambda) = 0$. There is a unitary map that maps a to v, and we can adjust its determinant to be 1. The conjugate of t by this map is a transvection in Q. Since $\mathrm{SU}_n(q)$ is generated by transvections, the normal closure of Q is the whole group. □

5 Simplicity

We can now apply Iwasawa's criterion to conclude that any normal subgroup of $\mathrm{SU}_n(q)$ is either contained in the kernel of its action on the polar space or contains the derived subgroup. By quotienting by the kernel, we get that $\mathrm{PSU}_n(q)$ is simple unless $(n, q) = \{(2, 2), (2, 3), (3, 2)\}$.

References

[1] Nick Gill. "Finite Classical Groups". `https://nickpgill.github.io/finite-classical-groups-2025`. Lecture notes for the London Taught Course Centre (LTCC). 2025.