Symplectic groups

Matevž Miščič

These notes are based on Lecture 4 of Nick Gill's course on Finite Classical Groups [1].

1 Isometries, similarities and semi-similarities

Let V be a vector space over a field $\mathbf{F}$ and let $\beta\colon V \times V \to \mathbf{F}$ be a σ-sesquilinear form on V.

Definition 1.1.
1. An *isometry* of (V, β) is a linear automorphism F of V such that $\beta(Fu, Fv) = \beta(u, v)$ for all $u, v \in V$.
2. A *similarity* of (V, β) is a linear automorphism F of V such that there is a $\lambda \in \mathbf{F}$ with $\beta(Fu, Fv) = \lambda\beta(u, v)$ for all $u, v \in V$.
3. A *semi-similarity* of (V, β) is a semi-linear automorphism F of V such that there is a $\lambda \in \mathbf{F}$ with $\beta(Fu, Fv) = \lambda\beta(u, v)$ for all $u, v \in V$.

Similarly we define isometries, similarities and semi-similarities of quadratic forms.

Definition 1.2. Let (V, β) be a formed space. A subspace $U \leq V$ is called *totally isotropic* if $\beta(u, v) = 0$ for all $u, v \in U$. A *polar space* associated with a formed space (V, β) is the set of all 1-dimensional totally isotropic subspaces of V.

We will often use the following lemma about extending isometries of subspaces to the whole space.

Theorem 1.3. *Witt's lemma Let (V, β) be a formed space and let $h\colon U_1 \to U_2$ be an isometry of two subspaces of V. Then h can be extended to an isometry of H of the whole space V.*

2 Forms and matrices

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V. Let $A \in \mathbf{F}^{n\times n}$ be the matrix defined by $A_{ij} = \beta(v_i, v_j)$. Then for all $u, v \in V$ we have

$$\beta(u, v) = u^T A v^\sigma,$$

where u and v on the right-hand side are the column vectors of the u and v with respect to $\mathcal{B}$.

Proposition 2.1. *We have*

1. *β is non-degenerate if and only if A is invertible.*
2. *β is symmetric if and only if $A = A^T$.*
3. *β is skew-symmetric if and only if $A = -A^T$.*
4. *β is σ-Hermitian if and only if $A^T = A^\sigma$.*

Proof. We only prove the last statement as the others are similar. The form β is σ-Hermitian if and only if $\beta(u,v) = \beta(v,u)^\sigma$ for all $u, v \in V$. This is equivalent to $u^T A v^\sigma = (v^T A u^\sigma)^\sigma = u^T A^{\sigma T} v^\sigma$, which is equivalent to $A^T = A^\sigma$. □

Proposition 2.2. *Let (V, β) be a formed space and let $\mathcal{B}$ be a basis of V. Suppose A is the matrix of β with respect to $\mathcal{B}$. Then the isometries of β are exactly those $g \in \mathrm{GL}(V)$ such that $g^T A g^\sigma = A$.*

Proof. Let g is an isometry of β. We have

$$u^T A v^\sigma = \beta(u,v) = \beta(gu, gv) = (gu)^T A (gv)^\sigma = u^T g^T A g^\sigma v^\sigma$$

for all $u, v \in V$. So g is an isometry if and only if $g^T A g^\sigma = A$. □

3 Symplectic groups

Let (V, β) be a symplectic formed space, that is, β is a non-degenerate alternating form on V. Last time we have seen that V is an orthogonal direct sum of hyperbolic lines, so there is a basis $\{e_1, f_1, \ldots, e_r, f_r\}$ of V such that

$$\beta(e_i, f_j) = \delta_{i,j}, \quad \beta(f_i, e_j) = -\delta_{i,j}, \quad \beta(e_i, e_j) = \beta(f_i, f_j) = 0.$$

By rearanging the basis to $\mathcal{B} = \{e_1, \ldots, e_r, f_r, \ldots, f_1\}$ the matrix of this form becomes

$$A = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}.$$

We define the followng groups:

1. $\mathrm{Sp}_{2r}(\mathbf{F})$ is the group of isometries of β.
2. $\mathrm{GSp}_{2r}(\mathbf{F})$ is the group of similarities of β.
3. $\Gamma\mathrm{Sp}_{2r}(\mathbf{F})$ is the group of semi-similarities of β.

We also define the projective versions of the groups above by taking the quotient by the subgroup of scalar transformations.

Lemma 3.1. *We have* $\mathrm{Sp}_2(\mathbf{F}) \cong \mathrm{SL}_2(\mathbf{F})$*.*

Proof. Define $\beta\colon \mathbf{F}^2 \times \mathbf{F}^2 \to \mathbf{F}$ by $\beta(u,v) = \det([u,v])$, where u, v are column vectors. Then β is a non-degenerate alternating form on $\mathbf{F}^2$ and $\mathrm{Sp}_2(\mathbf{F})$ contains exactly those $A \in \mathrm{GL}_2(\mathbf{F})$ that satisfy

$$\det([u,v]) = \beta(u,v) = \beta(Au,Av) = \det([Au,Av]) = \det(A)\det([u,v]).$$

These are exactly the matrices with determinant 1, so $\mathrm{Sp}_2(\mathbf{F}) = \mathrm{SL}_2(\mathbf{F})$. □

Proposition 3.2. *The group* $\mathrm{Sp}_{2r}(\mathbf{F})$ *acts primitively on its polar space.*

Proof. Let $G = \mathrm{Sp}_{2r}(\mathbf{F})$ and let Ω be the set of points of its polar space. By Witt's lemma, G acts transitively on Ω. Since any pair of distinct points of the polar space spans either a totally isotropic space or a hyperbolic line, G acts transitively on pairs of each type by Witt's lemma. It follows that G has at most three orbits on Ω^2, where the third orbit is the diagonal one. Any nontrivial congruence on Ω contains a diagonal orbit and exactly one of the other two. We claim that no congruence satisfies this.

Say that it contains all pairs of points that span a totally isotropic space. Let $u, v \in V$ be such that $\beta(u,v) \neq 0$. Then $u^\perp$ and $v^\perp$ are distinct hyperplanes of V, so there is $w \in V$ with $\beta(u,w) = \beta(v,w) = 0$. It follows $u \sim w \sim v$, so the congruence also contains all pairs of points that span a hyperbolic line. A similar argument works if the congruence contains all pairs of points that span a hyperbolic line. □

Proposition 3.3. *The point stabiliser* $G_{\langle e_1 \rangle}$ *has a normal subgroup* Q *isomorphic to the additive group* $\mathbf{F}$*.*

Proof. The point stabiliser $G_{\langle e_1 \rangle}$ contains all matrices of the block form

$$\begin{pmatrix} a & u^T & b \\ 0 & A & v \\ 0 & 0 & a^{-1} \end{pmatrix},$$

where $A \in \mathrm{Sp}_{2r-2}(\mathbf{F})$ and $u, v \in \mathbf{F}^{2r-2}$ with $u = (u_1, u_2)^T$ and $v = (-u_1, u_2)^T$ for some $u_1, u_2 \in \mathbf{F}^{r-1}$. Let us breafly explain why this is the case. Since the matrix maps e_1 to scalar multiple of itself, the first column is as it is. The matrix preserves the form, so it maps $e_1^\perp = \langle e_2, \ldots, e_r, f_1, \ldots, f_r \rangle$ to itself, which explains the last column.

This point stabiliser contains the subgroup of matrices $I + \lambda E_{1,2r}$ as a normal subgroup, so the proposition follows. □

4 Symplectic transvections

Recall that a *transvection* is a linear transformation t of V of the form $1+\mu$, where μ has rank 1 and $\mu^2 = 0$. We call a transvection t a *symplectic transvection* of the symplectic space (V, β) if t is an isometry of β.

Every transvection t is of the form $tv = v + f(v)a$, where $a \in V$ and $f \in V^*$ with $f(a) = 0$.

Lemma 4.1. *Every symplectic transvection is of the form $tv = v+\lambda\beta(v,a)a$ for some $\lambda \in \mathbf{F}$ and $a \in V$.*

Proof. The map t is a symplectic transvection if and only if $\beta(tv, tw) = \beta(v, w)$ for all $v, w \in V$. This is equivalent to

$$\beta(v, w) = \beta(v + f(v)a, w + f(w)a) = \beta(v, w) + f(v)\beta(a, w) + f(w)\beta(v, a).$$

By taking $w \in V$ such that $\beta(a, w) = 1$ and setting $\lambda = f(w)$, we see that $f(v) = \lambda\beta(v, a)$ for all v. The converse is easy to check. □

Lemma 4.2. *The symplectic transvections generate the symplectic group $\mathrm{Sp}_{2r}(\mathbf{F})$.*

Proof. Let $G = \mathrm{Sp}_{2r}(\mathbf{F})$ and let H be its subgroup generated by all transvections in G. We want to show that $H = G$. We first claim that H acts transitively on $V \setminus \{0\}$. Take $u, v \in V \setminus \{0\}$. If $\beta(u, v) \neq 0$, take $\lambda = \beta(u, v)^{-1}$ and $a = v-u$ and consider $t = 1+\lambda\beta(\,\cdot\,, a)a$. We have $tu = u+\beta(u, v)^{-1}\beta(u, v-u)(v-u) = v$. If $\beta(u, v) = 0$, there exists $w \in V \setminus \{0\}$ such that $\beta(u, w) \neq 0$ and $\beta(w, v) \neq 0$. By the argument from abouve, there is $h \in H$ with $hu = v$.

Next we claim that H is transitive on the set of hyperbolic pairs in V. By the previous claim it is enough to consider hyperbolic pairs (u, v) and (u, w). If $\beta(v, w) \neq 0$, it is easy to see that the transvection

$$x \mapsto x + \beta(v, w)^{-1}\beta(x, w - v)(w - v)$$

maps the first pair to the second. If $\beta(v, w) = 0$, we can go through the hyperbolic pair $(u, u + v)$ as before.

Now take an element $g \in G$. By the previous claim there is $h \in H$ such that $hge_1 = e_1$ and $hgf_1 = f_1$, so $hg \in \mathrm{Sp}_{2r-2}(\mathbf{F})$. By induction hg is a product of symplectic transvections, so the same holds for g. The base case follows from 3.1. □

Since all transvections have determinant 1, we have $\mathrm{Sp}_{2r}(\mathbf{F}) \leq \mathrm{SL}_{2r}(\mathbf{F})$.

Lemma 4.3. *The conjugacy class of every symplectic transvection intersects the group Q from Proposition 3.3.*

Proof. We know that in the symplectic basis $\{e_1, \ldots, e_r, f_r, \ldots, f_1\}$ the group Q has matrices of the form $I + \lambda E_{1,2r}$. Let t be a symplectic transvection such that $tv = v + \lambda\beta(v, a)a$. We can extend a to a hyperbolic pair (a, w) and then extend this to a symplectic basis $\{a, \ldots, w\}$ of V. With respect to this basis the matrix of t is $I - \lambda E_{1,2r}$. So if we conjugate t by the change of basis transformation, we get an element of Q. □

The previous two lemmas imply that $\langle Q^g \mid g \in \mathrm{Sp}_{2r}(\mathbf{F})\rangle = \mathrm{Sp}_{2r}(\mathbf{F})$.

Lemma 4.4. *Symplectic transvections in* $\mathrm{Sp}_{2r}(\mathbf{F})$ *are commutators except when* $(2r,|\mathbf{F}|) \in \{(2, 2), (2, 3), (4, 2)\}$.

Proof. Let $t\colon v \mapsto v + \lambda\beta(v, a)a$ be a symplectic transvection. Take $u \in V$ with $\beta(u, a) = 1$ and set $U = \langle u, a\rangle$. Then U is a hyperbolic line and t acts trivially on $U^{\perp}$. It follows that t is a transvection on U so it is a commutator in $\mathrm{SL}(U) = \mathrm{Sp}(U)$ provided that $|\mathbf{F}| > 3$. Thus it is also a commutator in $\mathrm{Sp}_{2r}(\mathbf{F})$ provided that $|\mathbf{F}| > 3$.

If $|\mathbf{F}| \leq 3$, then it is an easy calculation that in $\mathrm{Sp}_4(\mathbf{F}_3)$ and $\mathrm{Sp}_6(\mathbf{F}_2)$ every transvection is a commutator. The same then also holds in higher dimension by taking identity on the orthogonal complement. □

The previous lemma together with Lemma 4.2 implies that $\mathrm{Sp}_{2r}(\mathbf{F})$ is perfect except when $(2r,|\mathbf{F}|) \in \{(2, 2), (2, 3), (4, 2)\}$.

Since $\mathrm{PSp}_{2r}(\mathbf{F})$ also acts faithfully on the polar space, as every line is isotropic, we have everything to apply Iwasawa's criterion.

Theorem 4.5. *The group* $\mathrm{PSp}_{2r}(\mathbf{F})$ *is simple except when*

$$(2r,|\mathbf{F}|) \in \{(2, 2), (2, 3), (4, 2)\}.$$

We have $\mathrm{PSp}_2(\mathbf{F}) = \mathrm{PSL}_2(\mathbf{F}) \cong S_3$, $\mathrm{PSp}_2(3) = \mathrm{PSL}_2(3) \cong A_4$ and $\mathrm{PSp}_4(2) \cong S_6$.

References

[1] Nick Gill. "Finite Classical Groups". `https://nickpgill.github.io/finite-classical-groups-2025`. Lecture notes for the London Taught Course Centre (LTCC). 2025.