Linear Groups

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These notes are based on Lectures 1 and 2 of Nick Gill's course on Finite Classical Groups [1].

1 Semilinear transformations

Definition 1.1. Let V and W be vector spaces over a field $\mathbf{F}$. A *semilinear transformation from V to W* is a map $A\colon V \to W$ with the following properties:

1. $A(v + w) = Av + Aw$ for all $v, w \in V$.
2. There exists an automorphism σ of $\mathbf{F}$ such that $A(\lambda v) = \lambda^\sigma Av$ for all $\lambda \in \mathbf{F}$ and $v \in V$.

If $\sigma = \mathrm{id}$, then A is a linear transformation from V to W. We will be most interested in the case where $V = W$ and A is a bijection, in which case we call A a *semilinear automorphism of V*. We denote the set of all semilinear automorphisms of V by $\Gamma\mathrm{L}(V)$. This set is closed under composition, contains the identity map, and every element has an inverse, so $\Gamma\mathrm{L}(V)$ is a group under composition. We call $\Gamma\mathrm{L}(V)$ the *general semilinear group of V*.

Together with the *general linear group* and the *special linear group* we now have three important groups associated to a vector space V:

- The general semilinear group $\Gamma\mathrm{L}(V)$, consisting of all semilinear automorphisms of V.
- The general linear group $\mathrm{GL}(V)$, consisting of all linear automorphisms of V.
- The special linear group $\mathrm{SL}(V)$, consisting of all linear automorphisms of V with determinant 1.

There is an obvious inclusion $\mathrm{SL}(V) \le \mathrm{GL}(V) \le \Gamma\mathrm{L}(V)$.

Lemma 1.2. *The set of all invertible scalar transformations of V is a normal subgroup of $\Gamma\mathrm{L}(V)$.*

Proof. Let K be the set of all invertible scalar transformations of V. Clearly K is a subgroup of $\Gamma\mathrm{L}(V)$. Let $\lambda\,\mathrm{I} \in K$ be an invertible scalar transformation of V and let $A \in \Gamma\mathrm{L}(V)$ be a semilinear automorphism of V with associated field automorphism σ. Then for all $v \in V$ we have

$$A(\lambda\,\mathrm{I})A^{-1}v = A(\lambda A^{-1}v) = \lambda^\sigma v = (\lambda^\sigma\,\mathrm{I})v,$$

so $A(\lambda\,\mathrm{I})A^{-1} = \lambda^\sigma\,\mathrm{I}$ and thus K is normal in $\Gamma\mathrm{L}(V)$. □

Since the subgroup K from the previous lemma is normal in $\Gamma\mathrm{L}(V)$, it is also normal in $\mathrm{GL}(V)$, and $K \cap \mathrm{SL}(V)$ is normal in $\mathrm{SL}(V)$. We can thus form their quotients:

- The *projective semilinear group* $\mathrm{P}\Gamma\mathrm{L}(V) = \Gamma\mathrm{L}(V)/K$.
- The *projective linear group* $\mathrm{PGL}(V) = \mathrm{GL}(V)/K$.
- The *projective special linear group* $\mathrm{PSL}(V) = \mathrm{SL}(V)/(K \cap \mathrm{SL}(V))$.

The main goal of this lecture is to understand the structure of these groups and to show that the groups $\mathrm{PSL}(V)$ are simple for almost all finite vector spaces V.

2 Iwasawa's Criterion

We will prove the simplicity of the groups $\mathrm{PSL}(V)$ by studying their actions on the projective space $\mathbb{P}(V)$ of V or on the one dimensional subspaces of V. We will use the following criterion.

Theorem 2.1 (Iwasawa's Criterion). *Let G be a group acting primitively on a set Ω. Let $\omega \in \Omega$ and assume that G_ω has a normal abelian subgroup A such that*

$$G = \langle A^g \mid g \in G \rangle.$$

If $K \lhd G$, either $K \leq G_{(\Omega)}$ or $G' \leq K$. In particular, if G is perfect and faithful on Ω, then G is simple.

Proof. Let K be a normal subgroup of G that is not contained in $G_{(\Omega)}$. Since G acts primitively on Ω, it follows that K acts transitively on Ω and hence $G = G_\omega K$. Thus, for all $g \in G$, there exist $g' \in G_\omega$ and $k \in K$ such that $g = g'k$ and thus

$$\{A^g \mid g \in G\} = \{A^k \mid k \in K\}.$$

Now, since $G = \langle A^k \mid k \in K \rangle \leq A^K \leq G$ we conclude that $G = AK$. Then

$$G/K = AK/K \cong A/(A \cap K).$$

The right hand side is abelian, so G/K is as well. Hence, $G' \leq K$. □

3 Action on projective space

The group $\Gamma\mathrm{L}(V)$ acts on the set of points of the projective space $\mathbb{P}(V)$ of V by $A \cdot \langle v \rangle = \langle Av \rangle$ for all $A \in \Gamma\mathrm{L}(V)$ and $\langle v \rangle \in \mathbb{P}(V)$. This action is well defined, extends to an action on $P(V)$ and by the fundamental theorem of projective geometry, the group homomorphism $\Gamma\mathrm{L}(V) \to \mathrm{Aut}(\mathbb{P}(V))$ is surjective.

Lemma 3.1. *The action of* $\Gamma\mathrm{L}(V)$ *on* $\mathbb{P}(V)$ *has kernel* K *and thus induces a faithful action of* $\mathrm{P}\Gamma\mathrm{L}(V)$ *on* $\mathbb{P}(V)$.

Proof. Let $A \in \Gamma\mathrm{L}(V)$ be in the kernel of the action. Then for every $v \in V$ we have $Av \in \langle v \rangle$ and thus $Av = \lambda_v v$ for some $\lambda_v \in \mathbf{F}$. For linearly independent $u, v \in V$ we have

$$\lambda_u u + \lambda_v v = Au + Av = A(u+v) = \lambda_{u+v} u + \lambda_{u+v} v,$$

hence $\lambda_u = \lambda_{u+v} = \lambda_v$. If u and v are linearly dependent, then $\lambda_u = \lambda_w = \lambda_v$ for some $w \in V$ that is linearly independent of u (and thus also v). Hence, $A = \lambda\,\mathrm{I}$ for some $\lambda \in \mathbf{F}$. □

Similarly the action of $\mathrm{SL}(V)$ on $\mathbb{P}(V)$ induces a faithful action of $\mathrm{PSL}(V)$ on $\mathbb{P}(V)$.

Proposition 3.2. *The action of* $\mathrm{SL}(V)$ *on* $\mathbb{P}(V)$ *is* 2*-transitive.*

Proof. Take two pairs $\langle u_1 \rangle$, $\langle u_2 \rangle$ and $\langle v_1 \rangle$, $\langle v_2 \rangle$ of distinct points of $\mathbb{P}(V)$. We can extend $\{u_1, u_2\}$ to a basis $\{u_1, u_2, \dots, u_n\}$ of V and similarly extend $\{v_1, v_2\}$ to a basis $\{v_1, v_2, \dots, v_n\}$ of V. There exist a linear automorphism A of V such that $Au_1 = \alpha v_1$ and $Au_i = v_i$ for all $i \geq 1$. By choosing a suitable $\alpha \in \mathbf{F}$, we can ensure that $\det(A) = 1$. Thus, the action of $\mathrm{SL}(V)$ on $\mathbb{P}(V)$ is 2-transitive. □

Proposition 3.3. *Let* $G = \mathrm{SL}_n(q)$ *act on the set* $\Omega = \mathbb{P}(\mathbf{F}_q^n)$ *of points of the projective space. Then*

$$G_\omega \cong A \rtimes \mathrm{GL}_{n-1}(q),$$

where A *is isomorphis to the additive group of* $\mathbf{F}_q^{n-1}$.

Proof. Because of transitivity, we can assume that $\omega = \langle e_1 \rangle$. Then we have

$$G_\omega = \left\{ \begin{pmatrix} \lambda & v \\ 0 & A \end{pmatrix} \mid \lambda \in \mathbf{F}_q^*, v \in \mathbf{F}_q^{n-1}, A \in \mathrm{GL}_{n-1}(q), \det(A) = \lambda^{-1} \right\}.$$

Define $\phi\colon G_\omega \to \mathrm{GL}_{n-1}(q)$ that sends g to its lower right $(n-1) \times (n-1)$ block. Then ϕ is a surjective homomorphism with kernel A that is clearly isomorphic to the additive group of $\mathbf{F}_q^{n-1}$. It is easy to check that G is the semidirect product of A and $\mathrm{GL}_{n-1}(q)$. □

4 Transvections

Definition 4.1. Let V be a vector space over a field $\mathbf{F}$. A *transvection* of V is a linear endomorphism t of V such that $t-1$ has rank 1 and $(t-1)^2 = 0$.

Lemma 4.2. *Transvections of V lie in $\mathrm{SL}(V)$. They are conjugate in $\mathrm{GL}(V)$ and form at most two conjugacy classes in $\mathrm{SL}(V)$ all of which intersect A.*

Proof. Let t be a transvection. Let v_n be a vector not in $\ker(t-1)$ and let $v_1 = (t-1)v_n$. Clearly $v_1 \in \ker(t-1)$, so we can extend it to a basis $B = \{v_1, v_2, \ldots, v_n\}$ of $\ker(t-1)$. Then the matrix of t with respect to this basis is

$$\begin{pmatrix} 1 & & 1 \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Clearly the determinant is one and all transvections are conjugate in $\mathrm{GL}(V)$. If $\dim V \geq 3$, we can change v_2 to αv_2 to make the determinant of the transition matrix 1. In this case the transvections are also conjugate in $\mathrm{SL}(V)$. If $\dim V = 2$, then this change also changes the matrix of t to

$$\begin{pmatrix} 1 & & \alpha \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Clearly a conugate of each such matrix lies in A. $\square$

Lemma 4.3. *The group $\mathrm{SL}(V)$ is generated by the transvections of V.*

Proof. Let $A \in \mathrm{SL}(V)$. The matrices $I + \alpha E_{i,j}$ are all transvections for $i \neq j$ and $\alpha \in \mathbf{F}^*$. Multiplying any matrix by $I + \alpha E_{i,j}$ from the right adds α times the i-th column to the j-th column. Similarly multiplying it from the left adds α times the j-th row to the i-th row. By performing suitable row and column operations we can transform A to the identity, where the last diagonal will also be 1 because $\det(A) = 1$. Thus, A is a product of transvections. $\square$

Lemma 4.4. *Any transvection of V is a commutator, unless $\dim V = 2$ and $|\mathbf{F}| \leq 3$.*

Proof. If $|\mathbf{F}| \geq 3$, the following computation works for $n = 2$:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & (a^2-1)x \\ 0 & 1 \end{pmatrix}.$$

For $n \geq 3$ we can extend the above by identity on the complement of a 2 dimensional subspace.

A simple calculation works for $n \geq 3$ and $|\mathbf{F}| \in \{2, 3\}$. $\square$

The last two lemmas together show that $\mathrm{SL}(V)$ is perfect for all vector spaces V except when $\dim V = 2$ and $|\mathbf{F}| \leq 3$. The same is true also for $\mathrm{PSL}(V)$. The quotient image of a transvection in $\mathrm{PSL}(V)$ is called an *elation*.

We are now in position to apply Iwasawa's criterion.

Theorem 4.5. *The group* $\mathrm{PSL}(V)$ *is simple, except when* $\dim V = 2$ *and* $|\mathbf{F}| \leq 3$.

Proof. Let $G = \mathrm{PSL}(V)$ and let $\Omega = \mathbb{P}(V)$. We have seen that G acts faithfully and 2-transitively on Ω. In particular, the action is primitive. Let $\omega \in \Omega$ and let B be the quotient image of the subgroup A from Proposition 3.3 in G_ω. Then B is a normal abelian subgroup of G_ω. By Lemma 4.2 every elation is conjugate to an element of B and $\mathrm{PSL}(V)$ is generated by the elations, so $G = \langle B^g \mid g \in G \rangle$. Then Iwasawa's criterion applies and we conclude that G is simple. □

References

[1] Nick Gill. "Finite Classical Groups". `https://nickpgill.github.io/finite-classical-groups-2025`. Lecture notes for the London Taught Course Centre (LTCC). 2025.