Forms

Matevž Miščič

These notes are based on Lecture 3 of Nick Gill's course on Finite Classical Groups [1].

1 Sesquilinear forms

Let V be a vector space over a field $\mathbf{F}$.

Definition 1.1. A *sesquilinear form* on a vector space V is a map $\beta : V \times V \to \mathbf{F}$ such that for all $u \in V$ we have:

- The map $v \mapsto \beta(v, u)$ is linear.
- The map $v \mapsto \beta(u, v)$ is semilinear.

If the latter map is σ-sesquilinear, we say that β is a σ-sesquilinear form. For $\sigma = \mathrm{id}$ we say it is a bilinear form.

Definition 1.2. A sesquilinear form β is *reflexive* if $\beta(u, v) = 0$ implies $\beta(v, u) = 0$ for all $u, v \in V$.

A *radical* of a reflexive sesquilinear form β is the set

$$\mathrm{Rad}(\beta) = \{v \in V \mid \beta(v, u) = 0 \text{ for all } u \in V\}.$$

A reflexive sesquilinear form β is *non-degenerate* if its radical is $\{0\}$.

Let $\mathrm{PG}(V)$ be the set of all subspaces of V.

Definition 1.3. A *duality* induced by a sesquilinear form β is a map $\Delta : \mathrm{PG}(V) \to \mathrm{PG}(V)$ defined by

$$\Delta(U) = U^{\perp} = \{v \in V \mid \beta(v, u) = 0 \text{ for all } u \in U\}.$$

We say that a sesquilinear form β is

1. σ-Hermitian if $\beta(u, v) = \beta(v, u)^{\sigma}$ for all $u, v \in V$.
2. symmetric if $\beta(u, v) = \beta(v, u)$ for all $u, v \in V$.
3. alternating if $\beta(u, u) = 0$ for all $u \in V$.

4. skew-symmetric if $\beta(u,v) = -\beta(v,u)$ for all $u, v \in V$.

Lemma 1.4. *Let β be a non-degenerate sesquilinear form on V. Then*

1. *If β is σ-Hermitian, then $\sigma^2 = \mathrm{id}$ and $\beta(v,v) \in \mathrm{fix}(\sigma)$ for all $v \in V$.*

2. *If β is alternating, then it is also skew-symmetric.*

3. *If β is skew-symmetric and then $\mathrm{char}(\mathbf{F}) \neq 2$, then it is also alternating.*

4. *If β is symmetric, alternating or skew-symmetric, then it is bilinear.*

5. *If β is σ-Hermitian, symmetric, alternating or skew-symmetric, then it is reflexive.*

Theorem 1.5. *Let β be a non-degenerate reflexive σ-sesquilinear form on a vector space V of dimension at least 3. Then it is either symmetric, alternating or a scalar multiple of a σ-Hermitian form.*

Proof. We will only prove the theorem for the case when β is bilinear. We call a vector $u \in V$ *fine* if $\beta(u,v) = \beta(v,u) \neq 0$ for some $v \in V$.

1. Suppose there is no fine vector in V. Then $\beta(u,u) = 0$ for all $u \in V$, so β is alternating.

2. Suppose there is a fine vector $u \in V$ and let $v \in V$ be such that $\beta(u,v) = \beta(v,u) \neq 0$. We have the following calculation

$$\begin{aligned}
\beta(u,v)\beta(u,w) - \beta(u,w)\beta(u,v) &= 0 \\
\beta(u, \beta(u,v)w - \beta(u,w)v) &= 0 \\
\beta(\beta(u,v)w - \beta(u,w)v, u) &= 0 \\
\beta(u,v)\beta(w,u) &= \beta(v,u)\beta(u,w).
\end{aligned}$$

   We see that any fine vector commutes with all other vectors. So take any vector $w \in V$. If $\beta(u,w) \neq 0$, w is fine, so it commutes with every vector. If $\beta(u,w) = 0$, then $\beta(u, w+v) \neq 0$, so $w+v$ is fine. Since $w+v$ and v commute with every vector, so does w. We conclude that β is symmetric.

□

Definition 1.6. A σ-sesquilinear form β is *trace-valued* if $\beta(v,v) \in \mathrm{trace}(\sigma)$ for all $v \in V$, where

$$\mathrm{trace}(\sigma) = \{x + x^\sigma \mid x \in \mathbf{F}\}.$$

Lemma 1.7. *A non-degenerate σ-sesquilinear form β is not trace-valued if and only if $\mathrm{char}(\mathbf{F}) = 2$ and β is symmetric but not alternating.*

Proof. The lemma is trivial for alternating forms, so assume β is not alternating. We know that $\beta(u,u) \in \mathrm{fix}(\sigma)$ for all $u \in V$. We claim that $\mathrm{fix}(\sigma) = \mathrm{trace}(\sigma)$, which implies the lemma.

Let $x \in \mathrm{fix}(\sigma)$. If $\mathrm{char}(\mathbf{F}) \neq 2$ then $x = x/2 + (x/2)^\sigma \in \mathrm{trace}(\sigma)$. If $\mathrm{char}(\mathbf{F}) = 2$ and β is not symmetric, then there is a scalar $y \in \mathbf{F}$ such that $y \neq y^\sigma$. Thus $y + y^\sigma \neq 0$ and we have

$$x = \frac{xy}{y + y^\sigma} + \left(\frac{xy}{y + y^\sigma}\right)^\sigma \in \mathrm{trace}(\sigma).$$

□

2 Quadratic forms

Definition 2.1. A *quadratic form* on a vector space V is a map $Q : V \to \mathbf{F}$ such that for all $u, v \in V$ and $\lambda \in \mathbf{F}$ we have:

- $Q(\lambda u) = \lambda^2 Q(u)$.
- The map $\beta_Q : V \times V \to \mathbf{F}$ defined by $\beta_Q(u,v) = Q(u+v) - Q(u) - Q(v)$ is bilinear.

Definition 2.2. A *singular radical* of a quadratic form Q is the set

$$\mathrm{Rad}(Q) = \{v \in \mathrm{Rad}(\beta_Q) \mid Q(v) = 0\}.$$

A quadratic form Q is *non-singular* if its singular radical is $\{0\}$.

If $\mathrm{char}(\mathbf{F}) \neq 2$, there is a one-to-one correspondence between quadratic forms and symmetric bilinear forms. In one directon this correspondence is given by $Q \mapsto \beta_Q$, while in the other direction it is given by $\beta \mapsto Q_\beta$, where $Q_\beta(v) = \beta(v,v)/2$.

In $\mathrm{char}(\mathbf{F}) = 2$ many quadratic forms induce the same symmetric bilinear form.

3 Formed spaces

Definition 3.1. A *formed space* is a pair (V, κ), where V is a vector space and κ is either a trace-valued non-degenerate reflexive σ-sesquilinear form or a non-singular quadratic form on V.

The space (V, κ) is called

- *symplectic* if κ is an alternating bilinear form.
- *unitary* if κ is a σ-Hermitian form.

- *orthogonal* if κ is a symmetric bilinear form and $\mathrm{char}(\mathbf{F}) \neq 2$ or if κ is a quadratic form.

Definition 3.2. Formed spaces (V, Q) and (V', Q') are *isomorphic* if there is a linear isomorphism $A\colon V \to V'$ such that $Q'(Av) = Q(v)$ for all $v \in V$.

Formed spaces (V, β) and (V', β') are *isomorphic* if there is a linear isomorphism $A\colon V \to V'$ such that $\beta'(Au, Av) = \beta(u, v)$ for all $u, v \in V$.

Let (V, β) be a formed space and let $U \leq V$ be a subspace of V.

Definition 3.3. The subspace U is a *hyperbolic line* if $U = \langle u, v \rangle$ and $\beta(u, u) = \beta(v, v) = 0$ and $\beta(u, v) = 1$.

The subspace U is *anisotropic* if $\beta(u, u) \neq 0$ for all $u \in U \setminus \{0\}$.

Theorem 3.4. *A formed space (V, κ) is isomorphic to an orthogonal sum of hyperbolic lines and an anisotropic subspace.*

Proof. Define $f\colon V \to \mathbf{F}$ by $f(v) = \beta(v, v)$. If $f(u) \neq 0$ for all $u \neq 0$ then the whole space V is anisotropic and we are done. So assume $f(u) = 0$ for some $u \neq 0$. Since β is non-degenerate, there is a vector v such that $\beta(u, v) \neq 0$. We can scale v so that $\beta(u, v) = 1$. We have

$$f(v + \lambda u) = f(v) + \lambda + \lambda^\sigma,$$

so there is λ such that $f(v + \lambda u) = 0$, since β is trace-valued. We conclude that the subspace $\langle u, v + \lambda u \rangle$ is a hyperbolic line. We can now replace V with the orthogonal complement of this hyperbolic line and repeat the process. $\square$

We now want to determine the possible anisotropic spaces. If (V, β) is symplectic, the only possible anisotropic space is trivial, so V is an orthogonal sum of hyperbolic lines.

From now on we will assume that $\mathbf{F}$ is a finite field. Recall that for any prime power p^n there is a unique finite field $\mathbf{F}_{p^n}$ of order p^n and that $\mathrm{Aut}(\mathbf{F}_{p^n})$ is cyclic of order n, generated by the Frobenius automorphism $\phi\colon x \mapsto x^p$. For $\mathbf{F}_{p^n}$ to have an automorphism or order 2, n has to be even. In that case we can write $\mathbf{F}_{q^2}$ for the field and $\sigma\colon x \mapsto x^q$ for the automorphism of order 2.

Lemma 3.5. *Let (V, β) be a unitary formed space of dimension d and let U be an anisotropic subspace of V. Then $\dim U = 0$ if d is even and $\dim U = 1$ if d is odd. Also U is unique up to isomorphism.*

Proof. It is enough to prove that $\dim U \leq 1$. Assume that $\dim U \geq 2$ and let u, v be two orthogonal vectors in U. We have

$$\beta(u + \lambda v, u + \lambda v) = \beta(u, u) + \lambda\lambda^\sigma \beta(v, v).$$

There is $\lambda \in \mathbf{F}$ such that this is zero, which contradicts the fact that U is anisotropic.

Let $U = \langle u \rangle$ and $U' = \langle u' \rangle$ be two anisotropic subspaces. There exists $\lambda \in \mathbf{F}$ such that $\lambda\lambda^\sigma = \beta(u, u)/\beta(u', u')$. Then the linear isomorphism $A\colon U \to U'$ defined by $Au = \lambda u'$ satisfies $\beta'(Au, Au) = \beta(u, u)$. □

Lemma 3.6. *If* (V, Q) *is an orthogonal formed space and* U *is an anisotropic subspace of* V*, then* $\dim U \leq 2$*. Furthermore,* U *is unique up to isomorphism for each dimension, except when* $\mathrm{char}(\mathbf{F})$ *is odd and* $\dim U = 1$*. In this case there are two such subspaces, one is a non-square multiple of the other.*

Proof. Assume that $\dim U \geq 3$.

If $\mathrm{char}(\mathbf{F}) = 2$, then let u, v be two orthogonal vectors in U. Then $Q(u + \lambda v) = Q(u) + \lambda^2 Q(v)$. Since every element of $\mathbf{F}$ is a square, there is λ such that $Q(u + \lambda v) = 0$, which contradicts the fact that U is anisotropic.

If $\mathrm{char}(\mathbf{F})$ is odd, take three pairwise orthogonal vectors u, v, w in U. There exist $\lambda, \mu \in \mathbf{F}$ such that $\lambda Q(u) + \mu Q(v) + Q(w) = 0$. Then $Q(\lambda u + \mu v + w) = 0$, which contradicts the fact that U is anisotropic. □

References

[1] Nick Gill. “Finite Classical Groups”. `https://nickpgill.github.io/finite-classical-groups-2025`. Lecture notes for the London Taught Course Centre (LTCC). 2025.